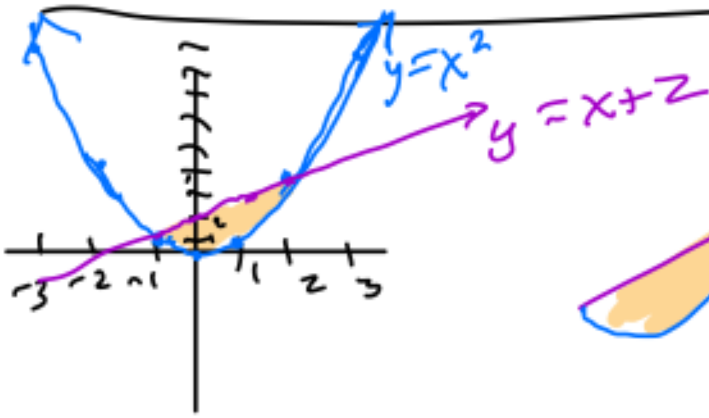
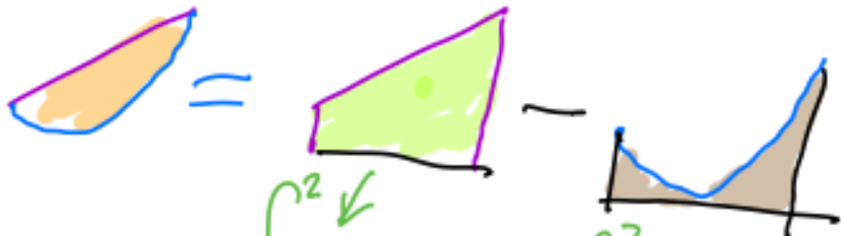
Example: Find the area between $y=x^2$ and $y=x+2$.





$$\int_{-1}^{2}(x+2)\,dx - \int_{-1}^{2} x^2\,dx$$

Check bounds:

$x+2 = x^2 \Rightarrow 0 = x^2 - x - 2 = (x+1)(x-2)$

$\Rightarrow x = -1$ or 2.

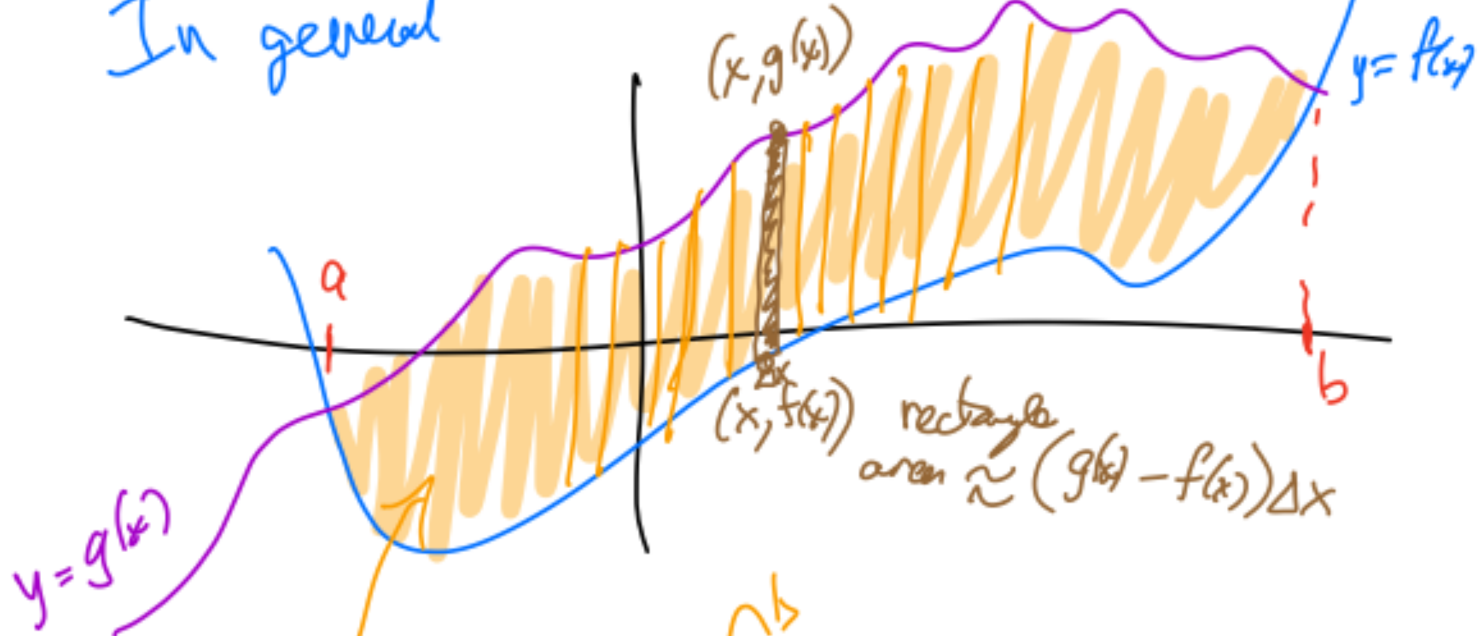
$$\Rightarrow \text{Area} = \int_{-1}^{2}(x+2)\,dx - \int_{-1}^{2} x^2\,dx$$

$$= \frac{x^2}{2}+2x\Big|_{-1}^{2} - \frac{x^3}{3}\Big|_{-1}^{2}$$

$$= \left(\frac{4}{2}+4\right) - \left(\frac{1}{2}+-2\right) - \left(\frac{8}{3} - \frac{(-1)}{3}\right)$$

$$= 6 - \left(\frac{-3}{2}\right) - \overset{\frac{9}{3}}{3} = 3+\frac{3}{2} = \boxed{\frac{9}{2}}.$$

In general



$$\text{Area} = \int_a^b \underbrace{(g(x) - f(x))\,dx}_{\text{area of each rectangular slice}}$$

Adds up over all the slices.

More examples of using substitution:

① $$\int \frac{2e^{2x}}{4+5e^{2x}}\,dx = \int \frac{du}{4+5u}$$

Let's try $u = e^{2x}$
$du = 2e^{2x}\,dx$

You could also choose $u = 4+5e^{2x}$
$du = 10e^{2x}\,dx$
$\frac{1}{10}du = e^{2x}\,dx$

$(\ln|x|)' = \frac{1}{x}$

$(\ln|4+5u|)' = \frac{1}{4+5u} \cdot 5$

$$= \frac{1}{5}\ln|4+5u| + C$$

$$= \boxed{\frac{1}{5}\ln|4+5e^{2x}| + C}$$

Check: $\left[\frac{1}{5}\ln|4+5u|+C\right]'$

$= \frac{1}{5}\left(\frac{1}{4+5u}\right)\cdot 5 = \frac{1}{4+5u}$ ✓

(2) $\displaystyle\int \frac{2}{4e^{-2x}+5}\,dx = \int \frac{2}{\frac{4}{e^{2x}}+5}\,dx$

try $u = 4e^{-2x}+5$

$du = -8e^{-2x}\,dx$

don't see it

$\displaystyle = \int \frac{2}{\frac{4+5e^{2x}}{e^{2x}}}\,dx$

$\displaystyle = \int \frac{2e^{2x}}{4+5e^{2x}}\,dx$

This #1

$\boxed{= \frac{1}{5}\ln|4+5e^{2x}|+C}$

(3) $\displaystyle\int \frac{e^{x}}{1+e^{2x}}\,dx =$

~~let $u = e^{2x}$~~

~~$du = 2e^{2x}\,dx$~~

~~don't see it~~

$\boxed{\text{let } u = e^{x} \quad du = e^{x}\,dx}$ $\Rightarrow u^2 = (e^x)^2 = e^{2x}$

$\displaystyle = \int \frac{du}{1+u^2}$

$$= \arctan(u) + C$$

$$= \boxed{\arctan(e^x) + C}$$

Check:

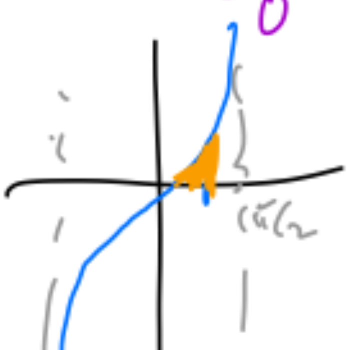
$$(\arctan(e^x))' = \frac{1}{1+(e^x)^2}\cdot e^x = \frac{e^x}{1+e^{2x}}.$$

Another possible guess:

$$[\ln(1+e^{2x})]' = \frac{1}{1+e^{2x}}\cdot 2e^{2x}$$

$$= \frac{2e^{2x}}{1+e^{2x}}$$ does not match.

4 $$\int_0^{\pi/4} \tan(\theta)\,d\theta = \cancel{\sec^2\theta}$$ Nope, need an anti-derivative



$$\int_0^{\pi/4} \frac{\sin(\theta)}{\cos(\theta)}\, d\theta$$

~~let $u = \sin\theta$~~

~~$du = \cos\theta\, d\theta$~~ ~~Need that in top~~

let $u = \cos\theta$

$du = -\sin\theta\, d\theta$

$-du = \sin\theta\, d\theta$

Bounds:

$\theta = 0 \quad u = \cos\theta = 1$

$\theta = \frac{\pi}{4} \Rightarrow u = \cos\theta = \frac{1}{\sqrt{2}}$

$$= \int_1^{1/\sqrt{2}} \frac{-du}{u} = -\int_1^{1/\sqrt{2}} \frac{1}{u}\, du$$

$$= -\ln(u)\Big|_1^{1/\sqrt{2}}$$

$$= -\ln\left(\frac{1}{\sqrt{2}}\right) - (-\underbrace{\ln(1)}_{0})$$

$$= -\ln\left(\frac{1}{\sqrt{2}}\right) = \ln(\sqrt{2}) = \ln(2^{1/2}) = \boxed{\frac{1}{2}\ln(2)}$$

$\ln(A^B) = B\ln A.$

$$\left(-\ln(\cos(\theta))\right)' = -\frac{1}{\cos\theta}\cdot -\sin\theta = \tan\theta$$

$$\left(-\ln|\cos(\theta)|\right)' = \tan\theta$$

(5) $\displaystyle\int \frac{x}{\sqrt{x^2+7}}\,dx$

Subs: let $u = x^2+7$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \int \frac{\frac{1}{2}du}{\sqrt{u}} = \int (\sqrt{u})'\,du$$

$$= \sqrt{u} + C$$

$$= \boxed{\sqrt{x^2+7} + C}$$

Alternatively,

$$\int \frac{\frac{1}{2}du}{\sqrt{u}} = \int \frac{1}{2} u^{-1/2} du$$

$$= \frac{1}{2} \frac{u^{-1/2+1}}{(-\frac{1}{2}+1)} + C$$

$$= \frac{1}{2} \frac{u^{1/2}}{(\frac{1}{2})} + C = u^{1/2} + C$$

$$= \boxed{(x^2+7)^{1/2} + C}$$

⑥ $\int \sqrt{3x+4}\, dx$

let $u = 3x+4$

$du = 3dx$

$\frac{1}{3}du = dx$

$= \int \sqrt{u}\left(\frac{1}{3}du\right) = \frac{1}{3}\int u^{1/2}\, du$

$= \frac{1}{3}\frac{u^{3/2}}{3/2} + C$

$= \frac{2}{9}u^{3/2} + C = \boxed{\frac{2}{9}(3x+4)^{3/2} + C}$

Check: $\left[\frac{2}{9}(3x+4)^{3/2} + C\right]' = \frac{\cancel{2}}{\cancel{9}} \cdot \frac{\cancel{3}}{\cancel{2}}(3x+4)^{1/2} \cdot \cancel{3}$

$= (3x+4)^{1/2}$ ✓

7. Dr Ri's Sneaky Trick of the Day 11/17

$$\int \sqrt{3\sqrt{x^3}+4}\, dx$$

let $u = 3\sqrt{x}+4$

$du = 3 \cdot \frac{1}{2}x^{-1/2}\, dx = \frac{3}{2\sqrt{x}}\, dx$

bad, not in the integral.

Sneaky solve for x, then take d

$u - 4 = 3\sqrt{x}$

$\frac{u-4}{3} = \sqrt{x} \Rightarrow x = \frac{(u-4)^2}{9}$

$dx = \frac{2(u-4)}{9}\, du$

$$= \int \sqrt{u} \left(\frac{2(u-4)}{9}\, du\right)$$

$$= \int \frac{u^{1/2}(2u-8)}{9}\,du = \frac{1}{9}\int \left(u^{1/2}(2u-8)\right)du$$

$$= \frac{1}{9}\int \left(2u^{3/2} - 8u^{1/2}\right)du$$

$$= \frac{1}{9}\left(\frac{2u^{5/2}}{5/2} - 8\frac{u^{3/2}}{3/2}\right) + C$$

$$= \frac{4}{45}u^{5/2} - \frac{8}{9}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{4}{45}u^{5/2} - \frac{16}{27}u^{3/2} + C$$

$$= \boxed{\frac{4}{45}\left(3\sqrt{x}+4\right)^{5/2} - \frac{16}{27}\left(3\sqrt{x}+4\right)^{3/2} + C}$$

Quiz

(1) What is your name, slightly misspelled?

(2) Favorite Thanksgiving food –
or other event

(3) Find the derivative

(a) e^x

(b) x

(c) $2x$

(4) Given the definition of

$$\int_a^b f(x)\,dx = \lim_{\;\to\;\;=\;} \sum f(\;)$$

where $= \frac{b-a}{n}$ and $= a + k\Delta x$